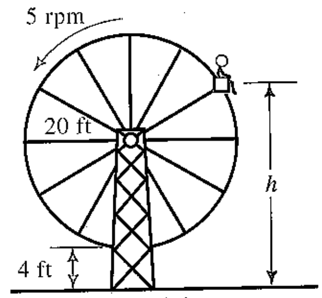
The Trigonometric Functions

7-3 The Sine and Cosine Functions

Objective *To use the definitions of sine and cosine to find values of these functions and to solve simple trigonometric equations.*

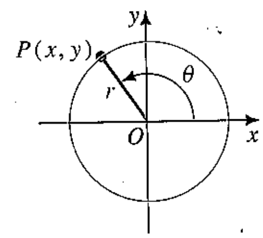
If you have ever ridden on a Ferris wheel, you may have wondered how to find your height above the ground at any given moment. Suppose a Ferris wheel has a radius of 20 ft and revolves at 5 rpm. If the bottom of the Ferris wheel sits 4 ft off the ground, then t seconds after the ride begins, a rider's height h above the ground is given in feet by:



$$h = 24 + 20 \sin(30t - 90)°$$

The "sin" that appears in the equation above is an abbreviation of the *sine* function, one of the two trigonometric functions that we will discuss in this section.

Suppose $P(x, y)$ is a point on the circle $x^2 + y^2 = r^2$ and θ is an angle in standard position with terminal ray OP, as shown at the right. We define the **sine** of θ, denoted $\sin \theta$, by:



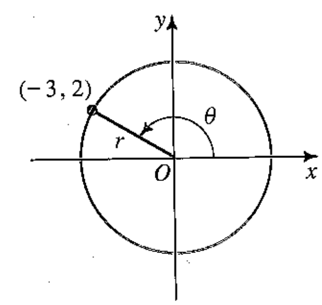
$$\sin \theta = \frac{y}{r}$$

and we define the **cosine** of θ, denoted $\cos \theta$, by:

$$\cos \theta = \frac{x}{r}$$

Example 1 If the terminal ray of an angle θ in standard position passes through $(-3, 2)$, find $\sin \theta$ and $\cos \theta$.

Solution Make a sketch as shown. To find the radius r of the circle, use the equation $x^2 + y^2 = r^2$ with $x = -3$ and $y = 2$:



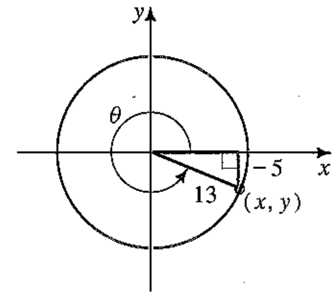
$$(-3)^2 + 2^2 = 13 = r^2$$
$$\sqrt{13} = r$$

Thus: $\sin \theta = \frac{y}{r} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$

and $\cos \theta = \frac{x}{r} = \frac{-3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}$

Example 2 If θ is a fourth-quadrant angle and $\sin\theta = -\frac{5}{13}$, find $\cos\theta$.

Solution Make a sketch of a circle with radius 13 as shown. Since $\sin\theta = \frac{y}{r} = -\frac{5}{13}$ and r is always positive, $y = -5$. To find x, use the circle's equation, $x^2 + y^2 = r^2$:



$$x^2 + (-5)^2 = 13^2$$
$$x^2 + 25 = 169$$
$$x^2 = 144$$
$$x = \pm 12$$

Since θ is a fourth-quadrant angle, $x = 12$. Thus, $\cos\theta = \frac{x}{r} = \frac{12}{13}$.

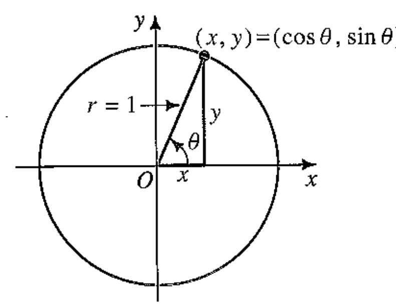
Although the definitions of $\sin\theta$ and $\cos\theta$ involve the radius r of a circle, the values of $\sin\theta$ and $\cos\theta$ depend only on θ, as the following activity shows.

Activity

You will need graph paper, a ruler, a compass, a protractor, and a calculator.

a. Using graph paper, draw an xy-plane and an acute angle θ in standard position.
b. Draw three concentric circles centered at the origin, and mark the points P_1, P_2, and P_3 where the circles intersect the terminal ray of θ.
c. Carefully measure the radii r_1, r_2, and r_3 of the three circles as well as the vertical distances y_1, y_2, and y_3 between P_1, P_2, and P_3 and the x-axis.
d. Use a calculator to compute $\frac{y_1}{r_1}$, $\frac{y_2}{r_2}$, and $\frac{y_3}{r_3}$ to the nearest hundredth. Each ratio is an approximation of the sine of θ. What do you observe about the ratios?
e. Use your knowledge of geometry to support your observation from part (d).

The circle $x^2 + y^2 = 1$ has radius 1 and is therefore called the **unit circle**. This circle is the easiest one with which to work because, as the diagram shows, $\sin\theta$ and $\cos\theta$ are simply the y- and x-coordinates of the point where the terminal ray of θ intersects the circle.



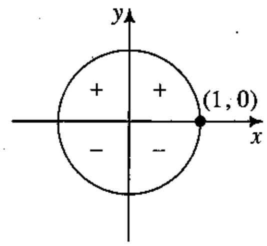
$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y$$

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x$$

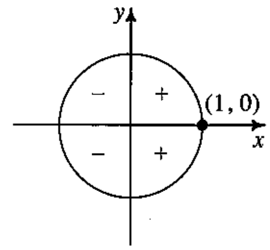
When a circle is used to define the trigonometric functions, they are sometimes called *circular functions*. (See Exercise 44 for another way to use the unit circle to define the trigonometric, or circular, functions.)

From the definitions and diagram at the bottom of the preceding page, we can see that the domain of both the sine and cosine functions is the set of all real numbers, since $\sin\theta$ and $\cos\theta$ are defined for any angle θ. Also, the range of both functions is the set of all real numbers between -1 and 1 inclusive, since $\sin\theta$ and $\cos\theta$ are the coordinates of points on the unit circle.

The diagrams below indicate where the sine and cosine functions have positive and negative values. For example, if θ is a second-quadrant angle, $\sin\theta$ is positive and $\cos\theta$ is negative.



$\sin\theta = y$

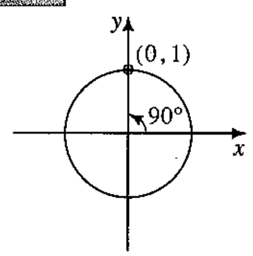


$\cos\theta = x$

Example 3 Find: **a.** $\sin 90°$ **b.** $\sin 450°$ **c.** $\cos(-\pi)$

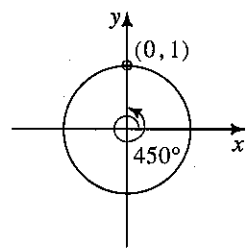
Solution

a.



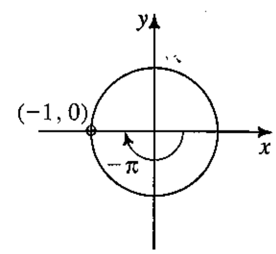
$\sin 90° = y\text{-coordinate} = 1$

b.



$\sin 450° = y\text{-coordinate} = 1$

c.



$\cos(-\pi) = x\text{-coordinate} = -1$

As figures (a) and (b) in Example 3 show, $\theta = 90°$ and $\theta = 450°$ are two solutions of the trigonometric equation $\sin\theta = 1$. The following example shows that there are infinitely many solutions of this equation.

Example 4 Solve $\sin\theta = 1$ for θ in degrees.

Solution You already know that $\theta = 90°$ is one solution of the equation $\sin\theta = 1$. Since any angle coterminal with 90° also has 1 as its sine value,

$$\theta = 90°,\ 90° \pm 360°,\ 90° \pm 2 \cdot 360°,\ 90° \pm 3 \cdot 360°,\ \ldots$$

are all solutions of the equation. They can be written more conveniently as $\theta = 90° + n \cdot 360°$, where n is an integer. (In radians, the solutions would be written as $\theta = \frac{\pi}{2} + n \cdot 2\pi$ or $\theta = \frac{\pi}{2} + 2n\pi$.)

From Example 4 and the definitions of $\sin \theta$ and $\cos \theta$, you can see that the sine and cosine functions repeat their values every 360° or 2π radians. Formally this means that for all θ:

$$\sin(\theta + 360°) = \sin \theta$$
$$\cos(\theta + 360°) = \cos \theta$$

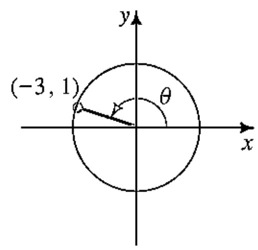
$$\sin(\theta + 2\pi) = \sin \theta$$
$$\cos(\theta + 2\pi) = \cos \theta$$

We summarize these facts by saying that the sine and cosine functions are *periodic* and that they have a *fundamental period* of 360°, or 2π radians. It is the periodic nature of these functions that makes them useful in describing many repetitive phenomena such as tides, sound waves, and the orbital paths of satellites.

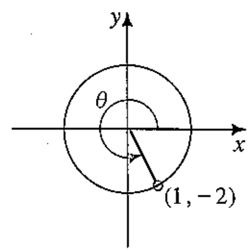
CLASS EXERCISES

Find $\sin \theta$ and $\cos \theta$.

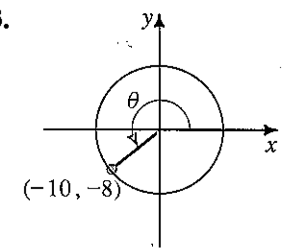
1.



2.



3.



4. State whether each expression is positive or negative.

a. $\sin 165°$ **b.** $\sin 265°$ **c.** $\cos 210°$ **d.** $\cos 310°$

e. $\sin \frac{5\pi}{6}$ **f.** $\cos \frac{5\pi}{6}$ **g.** $\sin \frac{4\pi}{3}$ **h.** $\cos \frac{5\pi}{3}$

i. $\sin 2$ **j.** $\cos 2$ **k.** $\sin 4$ **l.** $\cos 4$

5. Does $\cos \theta$ increase or decrease as:

a. θ increases from 0° to 90°? **b.** θ increases from 90° to 180°?

c. θ increases from 180° to 270°? **d.** θ increases from 270° to 360°?

6. Answer Exercise 5 for $\sin \theta$.

7. Use the unit circle to justify the fact that for all θ:

$$(\cos \theta)^2 + (\sin \theta)^2 = 1$$

8. There are infinitely many values of θ for which $\cos \theta = 0$. Name several.

9. a. Explain the meaning of $\theta = 45° + n \cdot 360°$, where n is an integer.

b. What is the equivalent statement if θ is expressed in radians?

WRITTEN EXERCISES

Find the value of each expression without using a calculator or table.

A

1. **a.** $\sin 180°$ **b.** $\cos 180°$ **c.** $\sin 270°$ **d.** $\cos 270°$
2. **a.** $\sin(-90°)$ **b.** $\cos(-90°)$ **c.** $\sin 360°$ **d.** $\cos 360°$
3. **a.** $\sin(-\pi)$ **b.** $\cos \pi$ **c.** $\sin \frac{3\pi}{2}$ **d.** $\cos \frac{\pi}{2}$
4. **a.** $\cos 2\pi$ **b.** $\sin\left(-\frac{\pi}{2}\right)$ **c.** $\sin 3\pi$ **d.** $\cos\left(-\frac{3\pi}{2}\right)$

Name each quadrant described.

5. **a.** $\sin\theta > 0$ and $\cos\theta < 0$ **b.** $\sin\theta < 0$ and $\cos\theta < 0$
6. **a.** $\sin\theta < 0$ and $\cos\theta > 0$ **b.** $\sin\theta > 0$ and $\sin(90° + \theta) > 0$

Without using a calculator or table, solve each equation for *all* θ in radians.

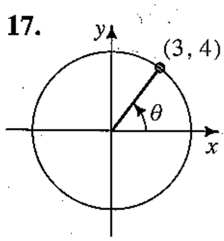
7. **a.** $\sin\theta = 1$ **b.** $\cos\theta = -1$ **c.** $\sin\theta = 0$ **d.** $\sin\theta = 2$
8. **a.** $\cos\theta = 1$ **b.** $\sin\theta = -1$ **c.** $\cos\theta = 0$ **d.** $\cos\theta = -3$

Without using a calculator or table, state whether each expression is positive, negative, or zero.

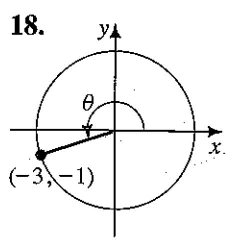
9. **a.** $\sin 4\pi$ **b.** $\cos\frac{7\pi}{6}$ **c.** $\sin\left(-\frac{\pi}{4}\right)$ **d.** $\cos\frac{3\pi}{4}$
10. **a.** $\cos 3\pi$ **b.** $\sin\frac{2\pi}{3}$ **c.** $\sin\frac{11\pi}{6}$ **d.** $\cos\left(-\frac{\pi}{2}\right)$
11. **a.** $\sin 60°$ **b.** $\cos(-120°)$ **c.** $\cos 300°$ **d.** $\sin(-210°)$
12. **a.** $\cos 45°$ **b.** $\sin 135°$ **c.** $\cos(-225°)$ **d.** $\sin(-315°)$
13. **a.** $\sin\frac{7\pi}{4}$ **b.** $\sin\left(-\frac{\pi}{6}\right)$ **c.** $\cos\frac{3\pi}{2}$ **d.** $\cos\frac{\pi}{3}$
14. **a.** $\cos\left(-\frac{\pi}{3}\right)$ **b.** $\sin\frac{\pi}{6}$ **c.** $\sin\frac{5\pi}{4}$ **d.** $\cos\frac{7\pi}{4}$
15. **a.** $\cos 89°$ **b.** $\cos 91°$ **c.** $\sin 720°$ **d.** $\sin(-270°)$
16. **a.** $\sin 1°$ **b.** $\sin(-1°)$ **c.** $\cos 90°$ **d.** $\cos 540°$

Find $\sin\theta$ and $\cos\theta$.

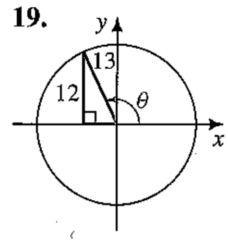
17.



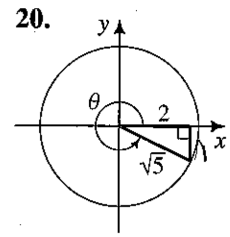
18.



19.



20.

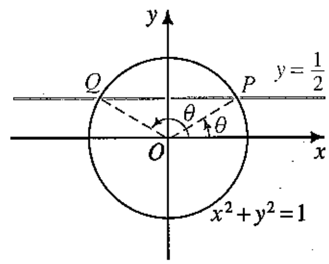


Complete the table. (A sketch like the one in Example 2 may be helpful.)

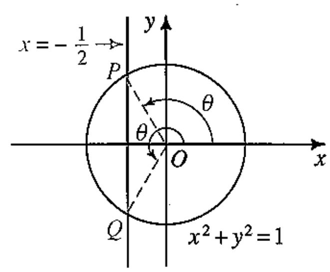
	21.	22.	23.	24.	25.	26.	27.	28.
Quadrant	I	II	III	IV	II	III	IV	II
$\sin\theta$	$\frac{3}{5}$	$\frac{5}{13}$	?	?	$\frac{1}{5}$	$-\frac{3}{7}$	?	$\frac{1}{9}$
$\cos\theta$	?	?	$-\frac{24}{25}$	$\frac{15}{17}$	?	?	$\frac{3}{4}$	?

B **29.** **a.** What are the coordinates of points P and Q where the line $y = \frac{1}{2}$ intersects the unit circle? (Refer to the diagram at the left below.)

b. Explain how part (a) shows that if $\sin\theta = \frac{1}{2}$, then $\cos\theta = \pm\frac{\sqrt{3}}{2}$.



Ex. 29



Ex. 30

30. **a.** What are the coordinates of points P and Q where the line $x = -\frac{1}{2}$ intersects the unit circle? (Refer to the diagram at the right above.)

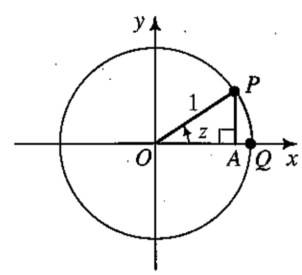
b. Explain how part (a) shows that if $\cos\theta = -\frac{1}{2}$, then $\sin\theta = \pm\frac{\sqrt{3}}{2}$.

31. ***Investigation*** In the diagram of the unit circle at the right, z is measured in radians.

a. Show that the length of arc PQ is z.

b. Show that the length of $\overline{PA}$ is $\sin z$.

c. What do parts (a) and (b) imply about the relationship between $\sin z$ and z for a small angle z? Confirm this relationship by using a calculator to compare $\sin z$ and z when z is a very small number of radians.



32. ***Investigation*** Refer to the diagram for Exercise 31.

a. Show that the length of $\overline{OA}$ is $\cos z$.

b. Use the results of part (c) of Exercise 31 and part (a) of this exercise to find an algebraic expression involving z (measure of the angle in radians) that approximates $\cos z$ for a small angle z.

c. Use a calculator to check the accuracy of the approximation in part (b) when z is a very small number of radians.

Without using a calculator or table, complete each statement with one of the symbols <, >, or =.

33. $\sin 40° \underline{\ ?\ } \sin 30°$

34. $\cos 40° \underline{\ ?\ } \cos 30°$

35. $\sin 172° \underline{\ ?\ } \sin 8°$

36. $\sin 310° \underline{\ ?\ } \sin 230°$

37. $\sin 130° \underline{\ ?\ } \sin 50°$

38. $\cos 50° \underline{\ ?\ } \cos (-50°)$

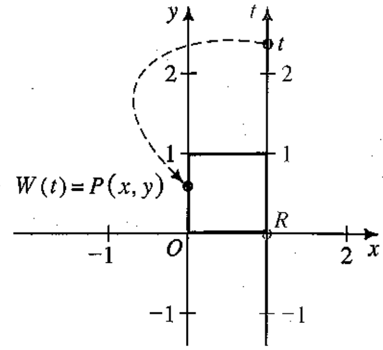
39. $\cos 214° \underline{\ ?\ } \cos 213°$

40. $\sin 169° \underline{\ ?\ } \sin 168°$

41. List in order of increasing size: $\sin 1$, $\sin 2$, $\sin 3$, $\sin 4$

42. List in order of increasing size: $\cos 1$, $\cos 2$, $\cos 3$, $\cos 4$

C **43.** Consider a special type of function called a *wrapping function*. This function, denoted by W, wraps a vertical number line whose origin is at $R(1, 0)$ around a unit square, as shown at the right. With each real number t on the vertical number line, W associates a point $P(x, y)$ on the square. For example, $W(1) = (1, 1)$ and $W(-1) = (0, 0)$. From W we can define two simpler functions:



$$c(t) = x\text{-coordinate of } P,$$

and $$s(t) = y\text{-coordinate of } P.$$

a. Find $W(2)$, $W(3)$, $W(4)$, and $W(5)$.

b. Explain why W is a periodic function and give its fundamental period.

c. Explain how the periodicity of W guarantees the periodicity of c and s.

d. Sketch the graphs of $u = c(t)$ and $u = s(t)$ in separate tu-planes.

44. ***Writing*** Suppose the unit square in Exercise 43 is replaced with the unit circle. Write a paragraph in which you describe how the wrapping function can now be used to define the circular functions sine and cosine.

//// COMPUTER EXERCISES

1. Use a computer to obtain the approximate value (to five decimal places) of

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

when $x = 1$, $x = 2$, and $x = \frac{\pi}{2} \approx 1.5708$. Compare the results with the values of SIN(1), SIN(2), and SIN(1.5708) given directly by the computer.

2. Use a computer to obtain the approximate value (to five decimal places) of

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

when $x = 1$, $x = 2$, and $x = \pi \approx 3.1416$. Compare the results with the values of COS(1), COS(2), and COS(3.1416) given directly by the computer.