ONE-TO-ONE FUNCTIONS AND THEIR INVERSES

Definitions:

- **One-to-one function**: is a function in which no two elements of the domain A have the same image. In other words, f is a one-to-one function if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- **Inverse function**: Let f be a one-to-one function with domain A and range B. Then its inverse function, denoted f^{-1}, has domain B and range A and is defined by

$$f^{-1}(y) = x \qquad \text{if and only if} \qquad f(x) = y$$

for any y in B.

Finding the inverse of a one-to-one function:

1. Replace $f(x)$ with y.
2. Interchange x and y.
3. Solve this equation for y. The resulting equation is $f^{-1}(x)$.

Important Properties:

- **Horizontal line test**: A function is one-to-one if no horizontal line intersects its graph more than once.
- **Property of inverse functions**: Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies

$$\begin{aligned} f^{-1}(f(x)) &= x \qquad \text{for every } x \text{ in } A \\ \text{and} \qquad f(f^{-1}(x)) &= x \qquad \text{for every } x \text{ in } B \end{aligned}$$

- The inverse of f^{-1} is f. So, we say that f and f^{-1} are inverses of each other.
- The inverse function interchanges the domain and range. Namely,

$$\text{Domain of } f = \text{Range of } f^{-1}$$
$$\text{Range of } f = \text{Domain of } f^{-1}$$

- The graph of f^{-1} is found by reflecting the graph of f across the line $y = x$.
- Only a one-to-one function can have an inverse.

Common Mistakes to Avoid:

- The -1 in the inverse f^{-1} is NOT an exponent. Be aware that

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

- In order for f to be a one-to-one function it must first be a function. Therefore, in order for f to be a one-to-one function it must pass both the vertical and horizontal line tests.

PROBLEMS

1. **Determine whether each function is a one-to-one function. (Remember f is one-to-one if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.)**

   (a) $f(x) = 8x - 3$

$$f(x_1) = f(x_2)$$
$$8x_1 - 3 = 8x_2 - 3$$
$$8x_1 = 8x_2$$
$$x_1 = x_2$$

$$\boxed{f \text{ is a one-to-one function}}$$

   (b) $f(x) = x^4 + 7$

$$f(x_1) = f(x_2)$$
$$x_1^4 + 7 = x_2^4 + 7$$
$$x_1^4 = x_2^4$$
$$\sqrt[4]{x_1^4} = \sqrt[4]{x_2^4}$$
$$x_1 = \pm x_2$$

$$\boxed{f \text{ is NOT a one-to-one function}}$$

2. **If f is a one-to-one function for which $f(1) = 7$, $f(-3) = 9$ and $f(6) = 2$ find $f^{-1}(9)$, $f^{-1}(7)$ and $f^{-1}(2)$.**

   Since f is a one-to-one function we know that it has an inverse. Remember that the inverse interchanges the x and y variable. Therefore,

$$\boxed{f^{-1}(9) = -3, \quad f^{-1}(7) = 1, \quad f^{-1}(2) = 6}$$

3. **Find the inverse of f.**

   (a) $f(x) = 3x - 5$

$$\begin{aligned} f(x) &= 3x - 5 \\ y &= 3x - 5 \\ x &= 3y - 5 \\ x + 5 &= 3y \\ \frac{x+5}{3} &= y \end{aligned}$$

$$\boxed{f^{-1}(x) = \frac{x+5}{3}}$$

   (b) $f(x) = 9 - 4x$

$$\begin{aligned} f(x) &= 9 - 4x \\ y &= 9 - 4x \\ x &= 9 - 4y \\ x + 4y &= 9 \\ 4y &= 9 - x \\ y &= \frac{9-x}{4} \end{aligned}$$

$$\boxed{f^{-1}(x) = \frac{9-x}{4}}$$

   (c) $f(x) = \dfrac{x-2}{6}$

$$\begin{aligned} f(x) &= \frac{x-2}{6} \\ y &= \frac{x-2}{6} \\ x &= \frac{y-2}{6} \\ 6x &= y - 2 \\ 6x + 2 &= y \end{aligned}$$

$$\boxed{f^{-1}(x) = 6x + 2}$$

(d) $f(x) = \dfrac{2}{x-4}$

$$f(x) = \frac{2}{x-4}$$
$$y = \frac{2}{x-4}$$
$$x = \frac{2}{y-4}$$
$$x(y-4) = 2$$
$$xy - 4x = 2$$
$$xy = 4x + 2$$
$$y = \frac{4x+2}{x}$$
$$\boxed{f^{-1}(x) = \frac{4x+2}{x}}$$

(e) $f(x) = \dfrac{x+1}{3x+2}$

$$f(x) = \frac{x+1}{3x+2}$$
$$y = \frac{x+1}{3x+2}$$
$$x = \frac{y+1}{3y+2}$$
$$x(3y+2) = y+1$$
$$3xy + 2x = y + 1$$
$$3xy = y + 1 - 2x$$
$$3xy - y = 1 - 2x$$
$$y(3x-1) = 1 - 2x$$
$$y = \frac{1-2x}{3x-1}$$
$$\boxed{f^{-1}(x) = \frac{1-2x}{3x-1}}$$

(f) $f(x) = x^2, \quad x \leq 0$

Note that with the restriction $x \leq 0$, the function $f(x) = x^2$ becomes a one-to-one function.

$$f(x) = x^2$$
$$y = x^2$$
$$x = y^2$$
$$\sqrt{x} = \sqrt{y^2}$$
$$\pm\sqrt{x} = y$$

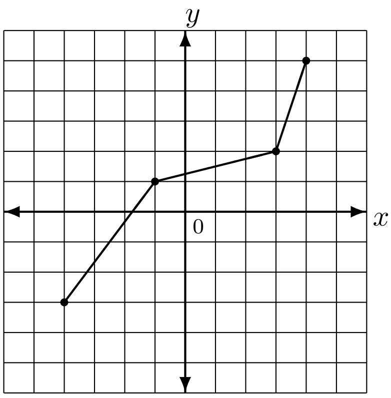
Now we need to decide whether our answer is $\sqrt{x}$ or $-\sqrt{x}$. Remember that the range of f^{-1} is the domain of f. Since the domain of f is $x \leq 0$ (negative numbers and zero), we need to choose $-\sqrt{x}$.

$$\boxed{f^{-1}(x) = -\sqrt{x}}$$

(g) $f(x) = \sqrt{4x-7}$

$$f(x) = \sqrt{4x-7}$$
$$y = \sqrt{4x-7}$$
$$x = \sqrt{4y-7}$$
$$x^2 = \left(\sqrt{4y-7}\right)^2$$
$$x^2 = 4y - 7$$
$$x^2 + 7 = 4y$$
$$\frac{x^2+7}{4} = y$$
$$\boxed{f^{-1}(x) = \frac{x^2+7}{4}, \ x \geq 0}$$

4. **Given the graph of f, sketch the graph of f^{-1}.**



To do this remember that the graph of f^{-1} is the reflection of f across the line $y = x$. Also, f^{-1} interchanges the x and y variables. Therefore, we will interchange the $x-$ and $y-$coordinates of each ordered pair. Once we graph these we will connect them with straight lines.